# ON CERTAIN GAMES WITH TRANSCENDENTAL VALUES

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P-188

November 13, 1950









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Let $\Gamma$ be a two person zero-sum game for which the compact pure strategy spaces, $S_1$ and $S_2$, and the payoff function $M$, defined over $S_1 \times S_2$, are definable in Tarski's system of "elementary algebra" (see [1]). Suppose, also, that $\Gamma$ has a value which is a transcendental number. We can then conclude that there is no optimal strategy for either player consisting of a step function of finitely many steps (i.e. a distribution in which the probabilities are all concentrated on a finite set of points). For, suppose the contrary for one of the players, say the maximizing one. Then, for some positive integer $m$, the value of $\Gamma$ is given by

$$v = \max_{\langle \alpha_1, \ldots, \alpha_m \rangle \in \mathcal{S}_m} \; \max_{x_1, \ldots, x_m \in S_1} \; \min_{y \in S_2} \sum_{i=1}^{m} \alpha_i M(x_i, y) ,$$

where $\mathcal{S}_m$ is the set of all m-tuples $\langle \mu_1, \ldots, \mu_m \rangle$ such that $\mu_i \geq 0$ for $i = 1, \ldots, m$, and $\sum_{i=1}^{m} \mu_i = 1$. But, according to [1], v would be algebraically definable, and it is a principal result of [1] that every algebraically definable number is algebraic.

In particular, our result applies to any game with transcendental value, in which M is a continuous rational function with integral coefficients.

**Example**: Take $M(x,y) = \dfrac{(1+x)(1-y)(1-xy)}{(1+xy)^2}$, $S_1 = \left[x \mid 0 \le x \le 1\right]$, and $S_2 = \left[y \mid 0 \le y \le 1\right]$, Here, $v = \frac{4}{\pi}$, and a pair of distribution functions yielding this value is given by:

$$\left.\begin{aligned} F^*(x) &= \frac{4}{\pi} \arctan \sqrt{x} \\ G^*(y) &= \frac{4}{\pi} \arctan \sqrt{y} \end{aligned}\right\} \; 0 \le {x \atop y} \le 1 .$$

Thus, in this game, there is no optimal strategy consisting of a step function of finitely many steps, for $\pi$ is a transcendental number.

## Reference

[1] Alfred Tarski. A decision method for elementary algebra and geometry. U.S. Air Force Project RAND, R-109. Prepared for publication by J.C.C. McKinsey. Lithoprinted. The RAND Corporation, Santa Monica, California, 1948, iii + 60 pp.